On the Castelnuovo-Mumford regularity of the cohomology of fusion systems and of the Hochschild cohomology of block algebras

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Abstract

Symonds' proof of Benson's regularity conjecture implies that the regularity of the cohomology of a fusion system and that of the Hochschild cohomology of a p-block of a finite group is at most zero. Using results of Benson, Greenlees, and Symonds, we show that in both cases the regularity is equal to zero.

Let p be a prime and k an algebraically closed field of characteristic p. Given a finite group G, a *block algebra of kG* is an indecomposable direct factor B of kG as a k-algebra. A *defect group of of a block algebra B of kG* is a minimal subgroup P of G such that B is isomorphic to a direct summand of $B \otimes_{kP} B$ as a B-B-bimodule. The defect groups of B form a G-conjugacy class of p-subgroups of G. The Hochschild cohomology of B is the algebra $HH^*(B) = \mathrm{Ext}^*_{B\otimes_k B^{\mathrm{op}}}(B)$, where B^{op} is the opposite algebra of B, and where B is regarded as a $B \otimes_k B^{\mathrm{op}}$-module via left and right multiplication. By a result of Gerstenhaber, the algebra $HH^*(B)$ is graded-commutative; that is, for homogeneous elements $\zeta \in HH^m(B)$ and $\eta \in HH^m(B)$ we have $\eta\zeta = (-1)^{nm}\zeta\eta$, where m, n are nonnegative integers. In particular, if $p = 2$, then $HH^*(B)$ is commutative, and if p is odd, then the even part $HH^{\mathrm{ev}}(B) = \oplus_{n\geq 0} HH^{2n}(B)$ is commutative and all homogeneous elements in odd degrees square to zero. The extension of the Castelnuovo-Mumford regularity to graded-commutative rings with generators in arbitrary positive degrees is due to Benson [2, §4]. We follow the notational conventions in Symonds [18]. In particular, if p is odd and $T = \oplus_{n\geq 0} T^n$ is a finitely generated graded-commutative k-algebra and M a finitely generated graded T-module, we denote by $\mathrm{reg}(T, M)$ the Castelnuovo-Mumford regularity of M as a graded T^{ev}-module, where $T^{\mathrm{ev}} = \oplus_{n\geq 0} T^{2n}$ is the even part of T. We set $\mathrm{reg}(T) = \mathrm{reg}(T, T)$; that is, $\mathrm{reg}(T)$ is the Castelnuovo-Mumford regularity of T as a graded T^{ev}-module. See also [3] and [8] for more background material and references. We note that Benson's definition of regularity uses the ring T instead of T^{ev}, but the two definitions are equivalent. This can be seen by noting that [18, Proposition 1.1] also holds for finitely generated graded commutative k-algebras.

Theorem 0.1 *Let G be a finite group and B a block algebra of kG. We have $\mathrm{reg}(HH^*(B)) = 0$.*

This will be shown as a consequence of a statement on Scott modules. Given a finite group G and a p-subgroup P of G, there is up to isomorphism a unique indecomposable kG-module $Sc(G;P)$ with vertex P and trivial source having a quotient (or equivalently, a submodule) isomorphic to the trivial kG-module k. The module $Sc(G;P)$ is called the *Scott module of kG with vertex P*. It is constructed as follows: Frobenius reciprocity implies that $\mathrm{Hom}_{kG}(\mathrm{Ind}_P^G(k), k) \cong \mathrm{Hom}_{kP}(k, k) \cong k$, and hence $\mathrm{Ind}_P^G(k)$ has up to isomorphism a unique direct summand $Sc(G;P)$ having k as a quotient. Since $\mathrm{Ind}_P^G(k)$ is selfdual, the uniqueness of $Sc(G;P)$ implies that $Sc(G;P)$ is also selfdual, and hence $Sc(G;P)$ can also be characterised as the unique summand, up to isomorphism, of $\mathrm{Ind}_P^G(k)$ having a nonzero trivial submodule. Moreover, it is not difficult to see that $Sc(G;P)$ has P has a vertex. See [7] for more details on Scott modules, as well as [11] for connections between Scott modules and fusion systems. For a finitely generated graded module X over $H^*(G;k)$ we denote by $H_m^{*,*}(X)$ the local cohomology with respect to the maximal ideal of $H^*(G;k)$ generated by all elements in positive degree. The first grading is here the local cohomological grading, and the second is induced by the grading of X.

Theorem 0.2 *Let G be a finite group and P a p-subgroup of G. We have*

$$\mathrm{reg}(H^*(G;k); H^*(G; Sc(G;P))) = 0 \ .$$

Remark 0.3 Using Benson's reinterpretation in [1, §4], of the 'last survivor' from [5, §7], applied to the Scott module instead of the trivial module, one can show more precisely that

$$H_m^{r,-r}(H^*(G; Sc(G,P))) \neq \{0\} \ ,$$

where r is the rank of P. It is not clear whether this property, or even the property of having cohomology with regularity zero, characterises Scott modules amongst trivial source modules.

For $\mathcal{F}$ a saturated fusion system on a finite p-group P, we denote by $H^*(P;k)^{\mathcal{F}}$ the graded subalgebra of $H^*(P;k)$ consisting of all elements ζ satisfying $\mathrm{Res}_Q^P(\zeta) = \mathrm{Res}_\varphi(\zeta)$ for any subgroup Q of P and any morphism $\varphi : Q \to P$ in $\mathcal{F}$. If $\mathcal{F}$ is the fusion system of a finite group G on one of its Sylow-p-subgroups P, then $H^*(P;k)^{\mathcal{F}}$ is isomorphic to $H^*(G;k)$ through the restriction map Res_P^G, by the characterisation of $H^*(G;k)$ in terms of stable elements due to Cartan and Eilenberg. In that case we have $\mathrm{reg}(H^*(P;k)^{\mathcal{F}}) = 0$ by [18, Corollary 0.2]. If $\mathcal{F}$ is the fusion system of a block algebra B of kG on a defect group P, then $H^*(P;k)^{\mathcal{F}}$ is the block cohomology $H^*(B)$ as defined in [14, Definition 5.1]. It is not known whether all block fusion systems arise as fusion systems of finite groups. There are examples of fusion systems which arise neither from finite groups nor from blocks; see [10], [13].

Theorem 0.4 *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. We have*

$$\mathrm{reg}(H^*(P;k)^{\mathcal{F}}) = 0 \ .$$

The key ingredients for proving the above results are Greenlees' local cohomology spectral sequence [9, Theorem 2.1], results and techniques in work of Benson [1], [2], [4], and Symonds' proof in [18] of Benson's regularity conjecture. We use the properties of the regularity from [18, §1] and [19, §2].

Lemma 0.5 *Let G be a finite group and V an indecomposable trivial source kG-module. Then* $\mathrm{reg}(H^*(G;k);H^*(G;V)) \leq 0$.

Proof Since V is a direct summand of $\mathrm{Ind}_P^G(k)$, we have

$$\mathrm{reg}(H^*(G;k);H^*(G;V)) \leq \mathrm{reg}(H^*(G;k);H^*(G;\mathrm{Ind}_P^G(k)) \ .$$

By [12, Lemma 4], the right side is equal to $\mathrm{reg}(H^*(P;k))$, hence zero by [18, Corollary 0.2]. □

Lemma 0.6 *Let G be a finite group and V a finitely generated kG-module. If $H_0(G;V) \neq \{0\}$, then* $\mathrm{reg}(H^*(G;k);H^*(G;V)) \geq 0$.

Proof It follows from the assumption $H_0(G;V) \neq \{0\}$ and Greenlees' spectral sequence [9, Theorem 2.1] that there is an integer s such that $H_m^{s,-s}(H^*(G;V)) \neq \{0\}$, which implies the result. □

Proof of Theorem 0.2 Set $V = Sc(G;P)$. By Lemma 0.5 we have

$$\mathrm{reg}(H^*(G;k);\mathrm{Ext}_{kG}^*(k;V)) \leq 0.$$

Since V has a nonzero trivial submodule, we have $H_0(G;V) \neq \{0\}$, and hence the other inequality follows from Lemma 0.6. □

Theorem 0.1 will be a consequence of Theorem 0.2 and the following well-known observation (for which we include a proof for the convenience of the reader; the block theoretic background material can be found in [20]).

Lemma 0.7 *Let G be a finite group, B a block algebra of kG and P a defect group of B. As a module over kG with respect to the conjugation action of G on B, the kG-module B has an indecomposable direct summand isomorphic to the Scott module $Sc(G;P)$.*

Proof Since the conjugation action of G on B induces the trivial action on $Z(B)$ and since $Z(B) \neq \{0\}$, it follows that the kG-module B has a nonzero trivial submodule. Moreover, B is a direct summand of kG, hence B is a p-permutation kG-module, and the vertices of the indecomposable direct summands of B are conjugate to subgroups of P. Thus B has a Scott module with a vertex contained in P as a direct summand. Since $Z(B)$ is not contained in the kernel of the Brauer homomorphism Br_P, it follows that B has a direct summand isomorphic to the Scott module $Sc(G;P)$. □

Proof of Theorem 0.1 By [12, Proposition 5] we have $\mathrm{reg}(HH^*(B)) \leq 0$. Recall that $HH^*(kG)$ is an $H^*(G;k)$-module via the diagonal induction map, and we have a canonical graded isomorphism $HH^*(B) \cong H^*(G;B)$ as $H^*(G;B)$-modules where G acts on B by conjugation; see e. g. [17, (3.2)]. It follows from [12, Lemma 4] that

$$\mathrm{reg}(HH^*(B)) = \mathrm{reg}(H^*(G;k);H^*(G;B)) \ .$$

By Lemma 0.7, the kG-module B has a direct summand isomorphic to $V = Sc(G;P)$, where P is a defect group of B. Thus as an $H^*(G;k)$-module, $H^*(G;B)$ has a direct summand isomorphic to $H^*(G;V)$. It follows that

$$\mathrm{reg}(HH^*(B)) \geq \mathrm{reg}(H^*(G;k); H^*(G;V)) = 0 ,$$

where the last equality is from Theorem 0.2. This completes the proof of Theorem 0.1. $\square$

Remark 0.8 The above proof can be adapted to show that the regularity of the stable quotient $\overline{HH^*}(B)$ of $HH^*(B)$ also equals zero. Recall that $\overline{HH^*}(B)$ is the quotient of $HH^*(B)$ by the ideal $Z^{\mathrm{pr}}(B) = \mathrm{Tr}_1^G(B)$ of $Z(B) \cong HH^0(B)$. Note that $Z^{\mathrm{pr}}(B)$ is concentrated in degree 0. Alternatively, $\overline{HH^*}(B)$ may be defined as the non-negative part of the Tate Hochschild cohomology of B. Our interest in $\overline{HH^*}(B)$ comes from the fact that Tate Hochschild cohomology of symmetric algebras is an invariant of stable equivalence of Morita type. We briefly indicate how the regularity of $\overline{HH^*}(B)$ may be calculated. Let $B = \oplus_i M_i$ be a decomposition of B into a direct sum of indecomposable kG-modules M_i, where G acts by conjugation on B. The canonical graded $H^*(G;k)$-module isomorphism $HH^*(B) \cong H^*(G;B)$ induces an isomorphism

$$HH^0(B) \cong H^0(G;B) = \oplus_i \ H^0(G;M_i)$$

in degree zero. Composing this with the the canonical isomorphisms $Z(B) \cong HH^0(B)$ and $H^0(G;M_i) \cong M_i^G$, it is easy to check that the image of $Z^{\mathrm{pr}}(B)$ in $\oplus_i M_i^G$ is $\oplus_i \mathrm{Tr}_1^G(M_i)$. Since B is a p-permutation kG-module, $\mathrm{Tr}_1^G(M_i)$ is non-zero precisely if M_i is isomorphic to the Scott module $Sc(G;1)$ (which is a projective cover of the trivial kG-module). Let M' denote the sum of all M_i's in the above decomposition which are isomorphic to $Sc(G,1)$ and let M'' be the complement of M' in B with respect to the above decomposition. Since $Z^{pr}(B)$ is concentrated in degree zero, we have a direct sum decomposition $HH^*(B) \cong \oplus H^*(G;M'') \oplus Z^{\mathrm{pr}}(B)$ as $H^*(G;k)$-modules. In particular,

$$\mathrm{reg}(H^*(G;k); HH^*(B)) = \max\{\mathrm{reg}(H^*(G;k); H^*(G;M'')), \mathrm{reg}(H^*(G;k); Z^{pr}(B))\}.$$

We may assume that a defect group P of B is non-trivial. By Lemma 0.7, M'' contains a direct summand isomorphic to $Sc(G;P)$. Hence by Theorem 0.2 $\mathrm{reg}(H^*(G;k); H^*(G;M'')) \geq 0$. It follows from Theorem 0.1 and the above displayed equation that $\overline{HH^*}(B) \cong H^*(G;M'')$ has regularity zero.

Proof of Theorem 0.4 By [18, Proposition 6.1] we have $\mathrm{reg}(H^*(P;k)^{\mathcal{F}}) \leq 0$. For the other inequality we follow the arguments in [1, §3, §4], applied to transfer maps using fusion stable bisets. For Q a subgroup of P and $\varphi : Q \to P$ an injective group homomorphism, we denote by $P \times_{(Q,\varphi)} P$ the P-P-biset of equivalence classes in $P \times P$ with respect to the relation $(uw, v) \sim (u, \varphi(w)v)$, where $u, v \in P$, and $w \in Q$. The kP-kP-bimodule having $P \times_{(Q,\varphi)} P$ as a k-basis is canonically isomorphic to $kP \otimes_{kQ} ({}_{\varphi}kP)$. This biset gives rise to a transfer map $\mathrm{tr}_{P\times_{(Q,\varphi)}P}$ on $H^*(P;k)$

obtained by composing the restriction map $\mathrm{res}^P_{\varphi(Q)} : H^*(P;k) \to H^*(\varphi(Q);k)$, the isomorphism $H^*(\varphi(Q);k) \cong H^*(Q;k)$ induced by φ, and the transfer map $\mathrm{tr}^P_Q : H^*(Q;k) \to H^*(P;k)$. Let X be an $\mathcal{F}$-stable P-P-biset satisfying the conclusions of [6, Proposition 5.5]. That is, every transitive subbiset of X is isomorphic to $P \times_{(Q,\varphi)} P$ for some subgroup Q of P and some group homomorphism $\varphi : Q \to P$ belonging to $\mathcal{F}$, the integer $|X|/|P|$ is prime to p, and for any subgroup Q of P and any group homomorphism $\varphi : Q \to P$ in $\mathcal{F}$, the Q-P-bisets ${}_\varphi X$ and ${}_Q X$ (resp. the P-Q-bisets X_Q and X_φ) are isomorphic. By taking the sum, over the transitive subbisets $P \times_{(Q,\varphi)} P$, of the transfer maps $\mathrm{tr}_{P\times_{(Q,\varphi)}P}$, we obtain a transfer map tr_X on $H^*(P;k)$. Following [15, Proposition 3.2], the map tr_X acts as multiplication by $\frac{|X|}{|P|}$ on $H^*(P;k)^{\mathcal{F}}$, hence $\mathrm{Im}(\mathrm{tr}_X) = H^*(P;k)^{\mathcal{F}}$, and we have a direct sum decomposition

$$H^*(P;k) = H^*(P;k)^{\mathcal{F}} \oplus \ker(\mathrm{tr}_X)$$

as $H^*(P;k)^{\mathcal{F}}$-modules. A similar decomposition holds for Tate cohomology, and for homology (using either the canonical duality $H_n(P;k) \cong H^n(P;k)^\vee$ or the isomorphism $H_n(P;k) \cong \hat{H}^{-n-1}(P;k)$ obtained from composing the previous duality with Tate duality). By [1, Equation (4.1)], the transfer map tr^P_Q induces a homomorphism of Greenlees' local cohomology spectral sequences

$$\begin{tikzcd}
H^{i,j}_m H^*(Q,k) \arrow[r, Rightarrow] \arrow[d, "(\mathrm{tr}^P_Q)_*"'] & H_{-i-j}(Q;k) \arrow[d, "(\mathrm{res}^P_Q)_*"] \\
H^{i,j}_m H^*(P;k) \arrow[r, Rightarrow] & H_{-i-j}(P;k)
\end{tikzcd}$$

where $(\mathrm{tr}^P_Q)_*$ and $(\mathrm{res}^P_Q)_*$ are the maps induced by tr^P_Q and the inclusion $Q \to P$, respectively. The isomorphism $\varphi : Q \to \varphi(Q)$ induces an obvious isomorphism of spectral sequences

$$\begin{tikzcd}
H^{i,j}_m H^*(\varphi(Q),k) \arrow[r, Rightarrow] \arrow[d, "\cong"'] & H_{-i-j}(\varphi(Q);k) \arrow[d, "\cong"] \\
H^{i,j}_m H^*(Q;k) \arrow[r, Rightarrow] & H_{-i-j}(Q;k)
\end{tikzcd}$$

Restriction and transfer on Tate cohomology are dual to each other under Tate duality, and hence the dual version of [1, Equation (4.1)] implies that the restriction $\mathrm{res}^P_{\varphi(Q)}$ induces a homomorphism of spectral sequences

$$\begin{tikzcd}
H^{i,j}_m H^*(P,k) \arrow[r, Rightarrow] \arrow[d, "(\mathrm{res}^P_{\varphi(Q)})_*"'] & H_{-i-j}(P;k) \arrow[d, "(\mathrm{tr}^P_{\varphi(Q)})_*"] \\
H^{i,j}_m H^*(\varphi(Q);k) \arrow[r, Rightarrow] & H_{-i-j}(\varphi(Q);k)
\end{tikzcd}$$

Composing the three diagrams above yields a homomorphism induced by $\mathrm{tr}_{P\times_{(Q,\varphi)}P}$ on the spectral sequence for P, and taking the sum over all transitive subbisets of

X yields a homomorphism of spectral sequences

$$\begin{tikzcd} H^{i,j}_m H^*(P,k) \arrow[r, Rightarrow] \arrow[d, "(\mathrm{tr}_X)_*"'] & H_{-i-j}(P;k) \arrow[d, "(\mathrm{tr}_{X^\vee})_*"] \\ H^{i,j}_m H^*(P;k) \arrow[r, Rightarrow] & H_{-i-j}(P;k) \end{tikzcd}$$

where $X^\vee$ is the P-P-biset X with the opposite action $u \cdot x \cdot v = v^{-1}xu^{-1}$ for all $u, v \in P$ and $x \in X$. One easily checks that $X^\vee$ is isomorphic to a dual basis of X in the dual bimodule $\mathrm{Hom}_k(kX, k)$. By [6, Proposition 5.2], $H^*(P;k)$ is finitely generated as a module over $H^*(P;k)^{\mathcal{F}}$. Thus the local cohomology spaces $H^{i,j}_m H^*(P;k)$ can be calculated using for m the maximal ideal of positive degree elements in $H^*(P;k)^{\mathcal{F}}$ instead of $H^*(P;k)$. It follows that tr_X induces a homomorphism of spectral sequences

$$\begin{tikzcd} H^{i,j}_m H^*(P,k) \arrow[r, Rightarrow] \arrow[d, "(\mathrm{tr}_X)_*"'] & H_{-i-j}(P;k) \arrow[d, "(\mathrm{tr}_{X^\vee})_*"] \\ H^{i,j}_m H^*(P;k)^{\mathcal{F}} \arrow[r, Rightarrow] & H_{-i-j}(P;k)^{\mathcal{F}} \end{tikzcd}$$

For $i = -j = r$, where r is the rank of P, the edge homomorphism yields a commutative diagram of the form

$$\begin{tikzcd} H^{r,-r}_m H^*(P;k) \arrow[r, "\gamma_P"] \arrow[d, "(\mathrm{tr}_X)_*"'] & H_0(P;k) \arrow[r, "\cong"] \arrow[d, "(\mathrm{tr}_{X^\vee})_*"'] & k \arrow[d, "\cdot\frac{|X|}{|P|}"] \\ H^{r,-r}_m H^*(P;k)^{\mathcal{F}} \arrow[r, "\delta_{\mathcal{F}}"'] & H_0(P;k)^{\mathcal{F}} \arrow[r, "\cong"'] & k \end{tikzcd}$$

where the right vertical map is multiplication on k by $\frac{|X|}{|P|}$. By [1, Theorem 4.1], the map γ_P is surjective, and hence so is the map $\delta_{\mathcal{F}}$. In particular, $H^{r,-r}_m H^*(P;k)^{\mathcal{F}} \neq \{0\}$, whence the result. $\square$

Remark 0.9 The fact that transfer and restriction on Tate cohomology are dual to each other under Tate duality can be deduced from a more general duality for transfer maps on Tate-Hochschild cohomology of symmetric algebras induced by bimodules which are finitely generated projective as left and right modules (cf. [16]).

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